RELAXATION AND 3D-2D PASSAGE WITH DETERMINANT TYPE CONSTRAINTS: AN OUTLINE

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Abstract. We outline our work (see [1, 2, 3, 4]) on relaxation and 3d-2d passage with determinant type constraints. Some open questions are addressed. This outline-paper comes as a companion to [5].

Contents

1. Relaxation with determinant type constraints

1.1. **Statement of the problem.**

Let $m, N \in \mathbb{N}$ (with $\min\{m,N\} > 1$), let $p > 1$ and let $W : \mathbb{M}^{m\times N} \to [0,+\infty]$ be Borel measurable and p-coercive, i.e.,

$$\exists C > 0\ \forall F \in \mathbb{M}^{m\times N}\ W(F) \geq C|F|^p,$$

where $\mathbb{M}^{m\times N}$ denotes the space of real $m \times N$ matrices. Define the functional $I : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,+\infty]$ by

$$I(\phi) := \int_\Omega W(\nabla\phi(x))dx,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, and consider $\overline{I} : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,+\infty]$ (the relaxed functional of I) given by

$$\overline{I}(\phi) := \inf\left\{\liminf_{n\to+\infty} I(\phi_n) : \phi_n \xrightarrow{L^p} \phi\right\}.$$

Denote the quasiconvex envelope of W by $\mathcal{Q}W : \mathbb{M}^{m\times N} \to [0,+\infty]$. The problem of the relaxation is the following:

($\mathcal{P}_1$) *prove (or disprove) that*

$$\forall \phi \in W^{1,p}(\Omega;\mathbb{R}^m) \ \ \overline{I}(\phi) = \int_\Omega \mathcal{Q}W(\nabla\phi(x))dx$$

and find a representation formula for $\mathcal{Q}W$.

At the begining of the eighties, Dacorogna answered to ($\mathcal{P}_1$) in the case where W is "finite and without singularities" (see §1.2). Recently, we extended the Dacorogna theorem as Theorem A and Theorem B (see §1.3 and §1.4) and we showed that these theorems can be used to deal with ($\mathcal{P}_1$) under the "weak-Determinant Constraint", i.e., when $m = N$ and $W : \mathbb{M}^{N\times N} \to [0,+\infty]$ is compatible with the following two conditions:

$$\text{(w-DC)} \begin{cases} W(F) = +\infty \iff -\delta \le \det F \le 0 \text{ with } \delta \ge 0 \text{ (possibly very large)} \\ W(F) \to +\infty \text{ as } \det F \to 0^+ \end{cases}$$

(see §1.6). However, the results of this section do not allow to treat ($\mathcal{P}_1$) under the "strong-Determinant Constraint", i.e., when $m = N$ and $W : \mathbb{M}^{N\times N} \to [0,+\infty]$ is compatible with the two basic conditions of nonlinear elasticity:

$$\text{(s-DC)} \begin{cases} W(F) = +\infty \iff \det F \le 0 & \text{(non-interpenetration of matter)} \\ W(F) \to +\infty \text{ as } \det F \to 0^+ & \left(\begin{array}{l}\text{necessity of an infinite amount} \\ \text{of energy to compress a finite} \\ \text{volume into zero volume}\end{array}\right) \end{cases}$$

(see §1.7).

1.2. Representation of $\mathcal{Q}W$ and $\overline{I}$: finite case.

Let $\mathcal{Z}_\infty W, \mathcal{Z}W : \mathbb{M}^{m\times N} \to [0,+\infty]$ be respectively defined by:

- ♦ $\mathcal{Z}_\infty W(F) := \inf\left\{\int_Y W(F + \nabla\varphi(y))dy : \varphi \in W_0^{1,\infty}(Y;\mathbb{R}^m)\right\}$;
- ♦ $\mathcal{Z}W(F) := \inf\left\{\int_Y W(F + \nabla\varphi(y))dy : \varphi \in \mathrm{Aff}_0(Y;\mathbb{R}^m)\right\}$,

where $Y :=]0,1[^N$, $W_0^{1,\infty}(Y;\mathbb{R}^m) := \{\varphi \in W^{1,\infty}(Y;\mathbb{R}^m) : \varphi = 0 \text{ on } \partial Y\}$ and $\mathrm{Aff}_0(Y;\mathbb{R}^m) := \{\varphi \in \mathrm{Aff}(Y;\mathbb{R}^m) : \varphi = 0 \text{ on } \partial Y\}$ with $\mathrm{Aff}(Y;\mathbb{R}^m)$ denoting the space of continuous piecewise affine functions from Y to $\mathbb{R}^m$.

Remark. One always has $W \ge \mathcal{Z}W \ge \mathcal{Z}_\infty W \ge \mathcal{Q}W$.

Theorem (Dacorogna [12] 1982)**.**

(a) *Representation of $\mathcal{Q}W$: if W is continuous and finite then*

$$\mathcal{Q}W = \mathcal{Z}W = \mathcal{Z}_\infty W.$$

(b) *Integral representation of $\overline{I}$: if W is continuous and*

$$\exists c > 0 \ \ \forall F \in \mathbb{M}^{m\times N} \ \ W(F) \le c(1 + |F|^p)$$

then
$$\forall \phi \in W^{1,p}(\Omega;\mathbb{R}^m) \quad \overline{I}(\phi) = \int_\Omega \mathcal{Q}W(\nabla\phi(x))dx.$$

1.3. Representation of $\mathcal{Q}W$: non-finite case.

The part (a) of the Dacorogna theorem can be extended as follows.

Theorem A (see [2, 3, 5])**.**

- ► *If $\mathcal{Z}_\infty W$ is finite then $\mathcal{Q}W = \mathcal{Z}_\infty W$.*
- ► *If $\mathcal{Z}W$ is finite then $\mathcal{Q}W = \mathcal{Z}W = \mathcal{Z}_\infty W$.*

Proof. We need (the two last assertions, the first one being used at the end of §1.3, of) the following result.

Theorem (Fonseca [16] 1988)**.**

1. *If $\mathcal{Z}_\infty W$ (resp. $\mathcal{Z}W$) is finite then $\mathcal{Z}_\infty W$ (resp. $\mathcal{Z}W$) is rank-one convex.*
2. *If $\mathcal{Z}_\infty W$ (resp. $\mathcal{Z}W$) is finite then $\mathcal{Z}_\infty W$ (resp. $\mathcal{Z}W$) is continuous.*
3. *$\mathcal{Z}_\infty W \le \mathcal{Z}\mathcal{Z}_\infty W$ and $\mathcal{Z}\mathcal{Z}W = \mathcal{Z}W$.*

One always has $W \ge \mathcal{Z}W \ge \mathcal{Z}_\infty W \ge \mathcal{Q}W$. Hence:

(i) $\mathcal{Q}\mathcal{Z}_\infty W = \mathcal{Q}W \le \mathcal{Z}_\infty W$;
(ii) $\mathcal{Q}\mathcal{Z}W = \mathcal{Q}\mathcal{Z}_\infty W = \mathcal{Q}W$.

► If $\mathcal{Z}_\infty W$ is finite then $\mathcal{Z}_\infty W$ is continuous by the property (2) of Fonseca. From the first part of the Dacorogna theorem it follows that $\mathcal{Q}\mathcal{Z}_\infty W = \mathcal{Z}\mathcal{Z}_\infty W$. But $\mathcal{Z}_\infty W \le \mathcal{Z}\mathcal{Z}_\infty W$ by the property (3) of Fonseca, and so $\mathcal{Q}W = \mathcal{Z}_\infty W$ by using (i).

► If $\mathcal{Z}W$ is finite then also is $\mathcal{Z}_\infty W$. Hence $\mathcal{Q}W = \mathcal{Z}_\infty W$ by the previous reasoning. On the other hand, $\mathcal{Z}W$ is continuous by the property (2) of Fonseca. From the first part of the Dacorogna theorem it follows that $\mathcal{Q}\mathcal{Z}W = \mathcal{Z}\mathcal{Z}W$. But $\mathcal{Z}\mathcal{Z}W = \mathcal{Z}W$ by the property (3) of Fonseca, and so $\mathcal{Q}W = \mathcal{Z}W$ by using (ii). □

Question. *Prove (or disprove) that if $\mathcal{Z}_\infty W$ is finite, also is $\mathcal{Z}W$.*

1.4. Representation of $\overline{I}$: non-finite case.

The part (b) of the Dacorogna theorem can be extended as follows.

Theorem B (see [2, 3, 5])**.**

- ► *If $\exists c > 0 \ \ \forall F \in \mathbb{M}^{m\times N} \quad \mathcal{Z}_\infty W(F) \le c(1+|F|^p)$ then*
$$\forall \phi \in W^{1,p}(\Omega;\mathbb{R}^m) \ \ \overline{I}(\phi) = \int_\Omega \mathcal{Q}W(\nabla\phi(x))dx.$$
- ► *If $\exists c > 0 \ \ \forall F \in \mathbb{M}^{m\times N} \quad \mathcal{Z}W(F) \le c(1+|F|^p)$ then*
$$\forall \phi \in W^{1,p}(\Omega;\mathbb{R}^m) \ \ \overline{I}(\phi) = \overline{I}_{\rm aff}(\phi) = \int_\Omega \mathcal{Q}W(\nabla\phi(x))dx$$
with $\overline{I}_{\rm aff} : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,+\infty]$ defined by
$$\overline{I}_{\rm aff}(\phi) := \inf\left\{\liminf_{n\to+\infty} I(\phi_n) : {\rm Aff}(\Omega;\mathbb{R}^m) \ni \phi_n \stackrel{L^p}{\to} \phi\right\}.$$

Outline of the proof. ► Let $\mathcal{Z}_\infty I, \overline{\mathcal{Z}_\infty I}, \overline{\mathcal{Z}_\infty I}_{\rm aff} : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,+\infty]$ be respectively defined by:

- ♦ $\mathcal{Z}_\infty I(\phi) := \displaystyle\int_\Omega \mathcal{Z}_\infty W(\nabla\phi(x))dx$;
- ♦ $\overline{\mathcal{Z}_\infty I}(\phi) := \inf\left\{\liminf_{n\to+\infty} \mathcal{Z}_\infty I(\phi_n) : \phi_n \stackrel{L^p}{\to} \phi\right\}$;

- ♦ $\overline{\mathcal{Z}_\infty I}_{\rm aff}(\phi) := \inf\left\{\liminf\limits_{n\to+\infty} \mathcal{Z}_\infty I(\phi_n) : {\rm Aff}(\Omega;\mathbb{R}^m) \ni \phi_n \xrightarrow{L^p} \phi\right\}$.

Since $\mathcal{Z}_\infty W$ is of p-polynomial growth, i.e., $\exists c>0\ \forall F\in\mathbb{M}^{m\times N}\ \mathcal{Z}_\infty W(F)\le c(1+|F|^p)$, it follows that $\mathcal{Z}_\infty W$ is (finite and so) continuous by the property (2) of Fonseca. By the second part of the Dacorogna theorem we deduce that

$$\forall \phi\in W^{1,p}(\Omega;\mathbb{R}^m)\ \ \overline{\mathcal{Z}_\infty I}(\phi) = \int_\Omega \mathcal{Q}\mathcal{Z}_\infty W(\nabla\phi(x))dx.$$

But one always has $\mathcal{Q}\mathcal{Z}_\infty W = \mathcal{Q}W$, hence

$$\forall \phi\in W^{1,p}(\Omega;\mathbb{R}^m)\ \ \overline{\mathcal{Z}_\infty I}(\phi) = \int_\Omega \mathcal{Q}W(\nabla\phi(x))dx.$$

Thus, it suffices to prove that $\overline{I}\le\overline{\mathcal{Z}_\infty I}$ (the reverse inequality being trivially true). The key point of the proof is that we can establish (by using the Vitali covering theorem and without assuming that $\mathcal{Z}_\infty W$ is of p-polynomial growth) the following lemma.

Lemma. $\overline{I}\le\overline{\mathcal{Z}_\infty I}_{\rm aff}$.

On the other hand, as $\mathcal{Z}_\infty W$ is of p-polynomial growth and ${\rm Aff}(\Omega;\mathbb{R}^m)$ is strongly dense in $W^{1,p}(\Omega;\mathbb{R}^m)$, it is easy to see that $\overline{\mathcal{Z}_\infty I}_{\rm aff} = \overline{\mathcal{Z}_\infty I}$, and the result follows.

► Let $\mathcal{Z}I, \overline{\mathcal{Z}_\infty I}, \overline{\mathcal{Z}_\infty I}_{\rm aff} : W^{1,p}(\Omega;\mathbb{R}^m)\to[0,+\infty]$ be respectively defined by:

- ♦ $\mathcal{Z}I(\phi) := \displaystyle\int_\Omega \mathcal{Z}W(\nabla\phi(x))dx$;
- ♦ $\overline{\mathcal{Z}I}(\phi) := \inf\left\{\liminf\limits_{n\to+\infty} \mathcal{Z}I(\phi_n) : \phi_n \xrightarrow{L^p} \phi\right\}$;
- ♦ $\overline{\mathcal{Z}I}_{\rm aff}(\phi) := \inf\left\{\liminf\limits_{n\to+\infty} \mathcal{Z}I(\phi_n) : {\rm Aff}(\Omega;\mathbb{R}^m) \ni \phi_n \xrightarrow{L^p} \phi\right\}$.

As $\mathcal{Z}W$ is of p-polynomial growth and (so) continuous (by the property (2) of Fonseca), from the second part of the Dacorogna theorem (and since $\mathcal{Q}\mathcal{Z}W=\mathcal{Q}W$ is always true) we deduce that

$$\forall \phi\in W^{1,p}(\Omega;\mathbb{R}^m)\ \ \overline{\mathcal{Z}I}(\phi) = \int_\Omega \mathcal{Q}W(\nabla\phi(x))dx.$$

It is then sufficient to prove that $\overline{I}_{\rm aff}\le\overline{\mathcal{Z}I}$ (the inequalities $\overline{I}\le\overline{I}_{\rm aff}$ and $\overline{\mathcal{Z}I}\le\overline{I}$ being trivially true). The key point of the proof is that we can establish (by using the Vitali covering theorem and without assuming that $\mathcal{Z}W$ is of p-polynomial growth) the following lemma.

Lemma. $\overline{I}_{\rm aff}=\overline{\mathcal{Z}I}_{\rm aff}$.

On the other hand, as $\mathcal{Z}W$ is of p-polynomial growth and ${\rm Aff}(\Omega;\mathbb{R}^m)$ is strongly dense in $W^{1,p}(\Omega;\mathbb{R}^m)$, it is clear that $\overline{\mathcal{Z}I}_{\rm aff}=\overline{\mathcal{Z}I}$, and the result follows. □

We see here that the integrands W for which $\mathcal{Z}_\infty W$ or $\mathcal{Z}W$ is of p-polynomial have a "nice" behavior with respect to $(\mathcal{P}_1)$. So, it could be interesting to introduce a new class of integrands (that we will call the class of p-ample[1] integrands) as follows:

$$W \text{ is } p\text{-ample} \iff \exists c>0\ \ \forall F\in\mathbb{M}^{m\times N}\ \ \mathcal{Z}_\infty W(F)\le c(1+|F|^p).$$

Thus, Theorems A and B can be summarized as follows.

Theorem A-B. *If W is p-ample then*

$$\forall \phi\in W^{1,p}(\Omega;\mathbb{R}^m)\ \ \overline{I}(\phi) = \int_\Omega \mathcal{Q}W(\nabla\phi(x))dx \text{ and } \mathcal{Q}W=\mathcal{Z}_\infty W.$$

[1]We use the term "p-ample" because of some analogies with the concept (developed in differential geometry by Gromov) of amplitude of a differential relation (see [17] for more details).

Question. *Prove (or disprove) that W is p-ample if and only if $\mathcal{Q}W$ is of p-polynomial growth.*

An analogue result of Theorem B was proved by Ben Belgacem (who is in fact the first that obtained an integral representation for $\overline{I}$ in the non-finite case). Let $\{\mathcal{R}_iW\}_{i\in\mathbb{N}}$ be defined by $\mathcal{R}_0W := W$ and for each $i \in \mathbb{N}^*$ and each $F \in \mathbb{M}^{m\times N}$,

$$\mathcal{R}_{i+1}W(F) := \inf_{\substack{a\in\mathbb{R}^N\\ b\in\mathbb{R}^m\\ t\in[0,1]}} \Big\{(1-t)\mathcal{R}_iW(F - ta\otimes b) + t\mathcal{R}_iW(F + (1-t)a\otimes b)\Big\}.$$

By Kohn et Strang (see [19]) we have $\mathcal{R}_{i+1}W \le \mathcal{R}_iW$ for all $i \in \mathbb{N}$ and $\mathcal{R}W = \inf_{i\ge 0}\mathcal{R}_iW$, where $\mathcal{R}W$ denotes the rank-one convex envelope of W. The Ben Belgacem theorem can be stated as follows.

Theorem (Ben Belgacem [8, 10] 1996)**.**
Assume that:

(BB$_1$) $\mathbb{O}_W := \mathrm{int}\Big\{F \in \mathbb{M}^{m\times N} : \forall i \in \mathbb{N}\ \mathcal{Z}\mathcal{R}_iW(F) \le \mathcal{R}_{i+1}W(F)\Big\}$ *is dense in* $\mathbb{M}^{m\times N}$;

(BB$_2$) $\forall i \in \mathbb{N}^*\quad \forall F \in \mathbb{M}^{m\times N}\quad \forall \{F_n\}_n \subset \mathbb{O}_W$

$$F_n \to F \Rightarrow \mathcal{R}_iW(F) \ge \limsup_{n\to+\infty} \mathcal{R}_iW(F_n);$$

- $\exists c > 0\quad \forall F \in \mathbb{M}^{m\times N}\quad \mathcal{R}W(F) \le c(1 + |F|^p)$.

Then

$$\forall \phi \in W^{1,p}(\Omega;\mathbb{R}^m)\quad \overline{I}(\phi) = \int_\Omega \mathcal{Q}\mathcal{R}W(\nabla\phi(x))dx.$$

Generally speaking, as rank-one convexity and quasiconvexity do not coincide, Theorem B and the Ben Belgacem theorem are not identical. However, we have

Lemma. *If either $\mathcal{Z}_\infty W$ or $\mathcal{Z}W$ is finite then $\mathcal{Q}\mathcal{R}W = \mathcal{Q}W$.*

Proof. If $\mathcal{Z}_\infty W$ (resp. $\mathcal{Z}W$) is finite then $\mathcal{Z}_\infty W$ (resp. $\mathcal{Z}W$) is rank-one convex by the property (1) of Fonseca. Consequently $\mathcal{Z}_\infty W \le \mathcal{R}W$ (resp. $\mathcal{Z}W \le \mathcal{R}W$) (and Theorem B$'$ below follows by applying Theorem B). Thus, we have $\mathcal{Z}_\infty W \le \mathcal{R}W \le W$ (resp. $\mathcal{Z}_\infty W \le \mathcal{R}W \le W$), hence $\mathcal{Q}\mathcal{Z}_\infty W \le \mathcal{Q}\mathcal{R}W \le \mathcal{Q}W$ (resp. $\mathcal{Q}\mathcal{Z}W \le \mathcal{Q}\mathcal{R}W \le \mathcal{Q}W$) and so $\mathcal{Q}\mathcal{R}W = \mathcal{Q}W$ since one always has $\mathcal{Q}\mathcal{Z}_\infty W = \mathcal{Q}W$ (resp. $\mathcal{Q}\mathcal{Z}W = \mathcal{Q}W$). □

Theorem B$'$. *Assume that $\exists c > 0\ \forall F \in \mathbb{M}^{m\times N}\ \mathcal{R}W(F) \le c(1 + |F|^p)$. Then*:

- *if $\mathcal{Z}_\infty W$ is finite then*

$$\forall \phi \in W^{1,p}(\Omega;\mathbb{R}^m)\quad \overline{I}(\phi) = \int_\Omega \mathcal{Q}W(\nabla\phi(x))dx;$$

- *if $\mathcal{Z}W$ is finite then*

$$\forall \phi \in W^{1,p}(\Omega;\mathbb{R}^m)\quad \overline{I}(\phi) = \overline{I}_{\mathrm{aff}}(\phi) = \int_\Omega \mathcal{Q}W(\nabla\phi(x))dx.$$

Question. *Prove (or disprove) that if* (BB$_1$) *and* (BB$_2$) *hold then $\mathcal{Z}W$ is finite.*

1.5. **Application 1: "non-zero-Cross Product Constraint".**
Consider $W_0 : \mathbb{M}^{3\times 2} \to [0, +\infty]$ Borel measurable and p-coercive and the following condition

$$\text{(P)}\ \exists \alpha, \beta > 0\ \forall \xi = (\xi_1 \mid \xi_2) \in \mathbb{M}^{3\times 2}\ \big(|\xi_1 \wedge \xi_2| \geq \alpha \Rightarrow W_0(\xi) \leq \beta(1 + |\xi|^p)\big)$$

with $\xi_1 \wedge \xi_2$ denoting the cross product of vectors $\xi_1, \xi_2 \in \mathbb{R}^3$. When W_0 satisfies (P) it is compatible with the "non-zero-Cross Product Constraint", i.e., with the following two conditions:

$$(*\text{-CPC}) \left\{ \begin{array}{l} W_0(\xi_1 \mid \xi_2) = +\infty \iff |\xi_1 \wedge \xi_2| = 0 \\ W_0(\xi_1 \mid \xi_2) \to +\infty \text{ as } |\xi_2 \wedge \xi_2| \to 0. \end{array} \right.$$

The interest of considering ($*$-CPC) comes from the 3d-2d problem (see §2): if W is compatible with (s-DC) then W_0 given by $W_0(\xi) := \inf_{\zeta \in \mathbb{R}^3} W(\xi \mid \zeta)$ is compatible with ($*$-CPC). One can prove that

$$\text{(P)} \Rightarrow \exists c > 0\ \forall F \in \mathbb{M}^{3\times 2}\ \mathcal{Z}W(F) \leq c(1 + |F|^p)$$

(see [2, 4, 5]) which roughly means that the "non-zero Cross Product Constraint" is p-ample. Applying Theorem B we obtain

Corollary 1. *If W_0 satisfies* (P) *then*

$$\forall \psi \in W^{1,p}(\Omega; \mathbb{R}^3)\ \ \overline{I}(\psi) = \overline{I}_{\text{aff}}(\psi) = \int_\Omega \mathcal{Q}W_0(\nabla\psi(x))dx.$$

1.6. **Application 2: "weak-Determinant Constraint".**
The following condition on W is compatible with (w-DC).

$$\text{(D)}\ \exists \alpha, \beta > 0\ \forall F \in \mathbb{M}^{N\times N}\ \big(|\text{det}F| \geq \alpha \Rightarrow W(F) \leq \beta(1 + |F|^p)\big).$$

One can prove that

$$\text{(D)} \Rightarrow \exists c > 0\ \forall F \in \mathbb{M}^{N\times N}\ \mathcal{Z}W(F) \leq c(1 + |F|^p)$$

(see [3, 5]) which roughly means that the "weak-Determinant Constraint" is p-ample. Applying Theorem B we obtain

Corollary 2. *If W satisfies* (D) *then*

$$\forall \phi \in W^{1,p}(\Omega; \mathbb{R}^N)\ \ \overline{I}(\phi) = \overline{I}_{\text{aff}}(\phi) = \int_\Omega \mathcal{Q}W(\nabla\phi(x))dx.$$

Proof of a part of Corollary 2. Taking the first part of Theorem B$'$ into account, it suffices to verify the following two points:

- ♦ (D) $\Rightarrow \exists c > 0\ \forall F \in \mathbb{M}^{N\times N}\ \mathcal{R}W(F) \leq c(1 + |F|^p)$;
- ♦ (D) $\Rightarrow \mathcal{Z}_\infty W < +\infty$,

which will give us the desired integral representation for $\overline{I}$. The first point is due to a lemma by Ben Belgacem (see [8], see also [5]). For the second point, it is obvious that $\mathcal{Z}_\infty W(F) < +\infty$ for all $F \in \mathbb{M}^{N\times N}$ with $|\text{det}F| \geq \alpha$. On the other hand, we have

Lemma (Dacorogna-Ribeiro [13] 2004, see also [11]).
$\forall F \in \mathbb{M}^{N\times N}\ \big(|\text{det}F| < \alpha \Rightarrow \exists \varphi \in W^{1,\infty}(Y; \mathbb{R}^N)\ \ |\text{det}(F + \nabla\varphi(x))| = \alpha\ \ \textit{p.p. dans } Y\big).$

Hence, if $F \in \mathbb{M}^{N\times N}$ is such that $|\text{det}F| < \alpha$ then $\mathcal{Z}_\infty W(F) \leq \int_Y W(F + \nabla\varphi(x))dx$ with some $\varphi \in W^{1,\infty}(Y; \mathbb{R}^N)$ given by the lemma above, and so $\mathcal{Z}_\infty W(F) \leq 2^p\beta(1 + |F|^p + \|\nabla\varphi\|^p_{L^p}) < +\infty$. □

1.7. **From p-ample to non-p-ample case.**
Because of the following theorem, none of the theorems of this section can be directly used for dealing with $(\mathcal{P}_1)$ under (s-DC).

Theorem (Fonseca [16] 1988)**.**
If W satisfies (s-DC) *then*:

- (F_1) $\mathcal{Q}W$ *is rank-one convex*;
- (F_2) $\mathcal{Q}W(F) = +\infty$ *if and only if* $\det F \le 0$ *and* $\mathcal{Q}W(F) \to +\infty$ *as* $\det F \to 0^+$.

The assertion (F_2) roughly says that the "strong-Determinant Constraint" is not p-ample, i.e., $\mathcal{Z}_\infty W$ cannot be of p-polynomial growth, and so neither Theorem A nor Theorem B is consistent with (s-DC). From the assertion (F_1) we see that $\mathcal{Q}W \le \mathcal{R}W$ which shows that $\mathcal{R}W$ cannot be of p-polynomial growth when combined with (F_2). Hence, the theorem of Ben Belgacem is not compatible with (s-DC).

Question. *Develop strategies for passing from p-ample to non-p-ample case.*

2. 3D-2D passage with determinant type constraints

2.1. **Statement of the problem.**
Let $W : \mathbb{M}^{3\times 3} \to [0, +\infty]$ be Borel measurable and p-coercive (with $p > 1$) and, for each $\varepsilon > 0$, let $I_\varepsilon : W^{1,p}(\Sigma_\varepsilon; \mathbb{R}^3) \to [0, +\infty]$ be defined by

$$I_\varepsilon(\phi) := \frac{1}{\varepsilon}\int_{\Sigma_\varepsilon} W(\nabla\phi(x, x_3))dxdx_3,$$

where $\Sigma_\varepsilon := \Sigma\times]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[\subset \mathbb{R}^3$ with $\Sigma \subset \mathbb{R}^2$ Lipschitz, open and bounded, and a point of Σ_ε is denoted by (x, x_3) with $x \in \Sigma$ and $x_3 \in]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[$. The problem of 3d-2d passage is the following.

$(\mathcal{P}_2)$ *Prove (or disprove) that*

$$\forall\psi \in W^{1,p}(\Sigma; \mathbb{R}^3) \quad \Gamma(\pi)\text{-}\lim_{\varepsilon\to 0} I_\varepsilon(\phi) = \int_\Sigma W_{\rm mem}(\nabla\psi(x))dx$$

and find a representation formula for $W_{\rm mem} : \mathbb{M}^{3\times 2} \to [0, +\infty]$.

At the begining of the nineties, Le Dret and Raoult answered to $(\mathcal{P}_2)$ in the case where W is "finite and without singularities" (see §2.3). Recently, we extended the Le Dret-Raoult theorem to the case where W is compatible with (w-DC) and (s-DC) as Theorem C and Theorem D (see §2.4 and §2.5).

2.2. **The $\Gamma(\pi)$-convergence.**
The concept of $\Gamma(\pi)$-convergence was introduced Anzellotti, Baldo and Percivale in order to deal with dimension reduction problems in mechanics. Let $\pi = \{\pi_\varepsilon\}_\varepsilon$ be the family of L^p-continuous maps $\pi_\varepsilon : W^{1,p}(\Sigma_\varepsilon; \mathbb{R}^3) \to W^{1,p}(\Sigma; \mathbb{R}^3)$ defined by

$$\pi_\varepsilon(\phi) := \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \phi(\cdot, x_3)dx_3.$$

Definition (Anzellotti-Baldo-Percivale [6] 1994)**.**
We say that $\{I_\varepsilon\}_\varepsilon$ $\Gamma(\pi)$-converge to $I_{\rm mem}$ as ε goes to zero, and we write

$$I_{\rm mem} = \Gamma(\pi)\text{-}\lim_{\varepsilon\to 0} I_\varepsilon,$$

if and only if

$$\forall\psi\in W^{1,p}(\Sigma;\mathbb{R}^3)\quad \left(\Gamma(\pi)\text{-}\liminf_{\varepsilon\to 0} I_\varepsilon\right)(\psi)=\left(\Gamma(\pi)\text{-}\limsup_{\varepsilon\to 0} I_\varepsilon\right)(\psi)=I_{\rm mem}(\psi)$$

with $\Gamma(\pi)\text{-}\liminf\limits_{\varepsilon\to 0} I_\varepsilon, \Gamma(\pi)\text{-}\limsup\limits_{\varepsilon\to 0} I_\varepsilon : W^{1,p}(\Sigma;\mathbb{R}^3)\to[0,+\infty]$ *respectively given by:*

- ♦ $\Gamma(\pi)\text{-}\liminf\limits_{\varepsilon\to 0} I_\varepsilon(\psi):=\inf\left\{\liminf\limits_{\varepsilon\to 0} I_\varepsilon(\phi_\varepsilon):\pi_\varepsilon(\phi_\varepsilon)\xrightarrow{L^p}\psi\right\}$;
- ♦ $\Gamma(\pi)\text{-}\limsup\limits_{\varepsilon\to 0} I_\varepsilon(\psi):=\inf\left\{\limsup\limits_{\varepsilon\to 0} I_\varepsilon(\phi_\varepsilon):\pi_\varepsilon(\phi_\varepsilon)\xrightarrow{L^p}\psi\right\}$.

Anzellotti, Baldo and Percivale proved that their concept of $\Gamma(\pi)$-convergence is not far from that of Γ-convergence introduced by De Giorgi and Franzoni. For each $\varepsilon>0$, consider $\mathcal{I}_\varepsilon : W^{1,p}(\Sigma;\mathbb{R}^3)\to[0,+\infty]$ defined by

$$\mathcal{I}_\varepsilon(\psi):=\inf\Big\{I_\varepsilon(\phi):\pi_\varepsilon(\phi)=\psi\Big\}.$$

Definition (De Giorgi-Franzoni [15, 14] 1975)**.**
We say that $\{\mathcal{I}_\varepsilon\}_\varepsilon$ Γ*-converge to* $I_{\rm mem}$ *as* ε *goes to zero, and we write*

$$I_{\rm mem}=\Gamma\text{-}\lim_{\varepsilon\to 0}\mathcal{I}_\varepsilon,$$

if and only if

$$\forall\psi\in W^{1,p}(\Sigma;\mathbb{R}^3)\quad \left(\Gamma\text{-}\liminf_{\varepsilon\to 0}\mathcal{I}_\varepsilon\right)(\psi)=\left(\Gamma\text{-}\limsup_{\varepsilon\to 0}\mathcal{I}_\varepsilon\right)(\psi)=I_{\rm mem}(\psi)$$

with $\Gamma\text{-}\liminf\limits_{\varepsilon\to 0}\mathcal{I}_\varepsilon, \Gamma\text{-}\limsup\limits_{\varepsilon\to 0}\mathcal{I}_\varepsilon : W^{1,p}(\Sigma;\mathbb{R}^3)\to[0,+\infty]$ *respectively given by:*

- ♦ $\Gamma\text{-}\liminf\limits_{\varepsilon\to 0}\mathcal{I}_\varepsilon(\psi):=\inf\left\{\liminf\limits_{\varepsilon\to 0}\mathcal{I}_\varepsilon(\psi_\varepsilon):\psi_\varepsilon\xrightarrow{L^p}\psi\right\}$;
- ♦ $\Gamma\text{-}\limsup\limits_{\varepsilon\to 0}\mathcal{I}_\varepsilon(\psi):=\inf\left\{\limsup\limits_{\varepsilon\to 0}\mathcal{I}_\varepsilon(\psi_\varepsilon):\psi_\varepsilon\xrightarrow{L^p}\psi\right\}$.

The link between $\Gamma(\pi)$-convergence and Γ-convergence is given by the following lemma.

Lemma (see [6])**.**
$I_{\rm mem}=\Gamma(\pi)\text{-}\lim\limits_{\varepsilon\to 0} I_\varepsilon$ *if and only if* $I_{\rm mem}=\Gamma\text{-}\lim\limits_{\varepsilon\to 0}\mathcal{I}_\varepsilon$.

2.3. $\Gamma(\pi)$**-convergence of** I_ε**: finite case.**
Let $W_0:\mathbb{M}^{3\times 2}\to[0,+\infty]$ be defined by

$$W_0(\xi):=\inf_{\zeta\in\mathbb{R}^3} W(\xi\mid\zeta).$$

Theorem (Le Dret-Raoult [20, 21] 1993)**.**
If W *is continuous and* $\exists c>0$ $\forall F\in\mathbb{M}^{3\times 3}$ $W(F)\le c(1+|F|^p)$ *then*

$$\forall\psi\in W^{1,p}(\Sigma;\mathbb{R}^3)\ \ \Gamma(\pi)\text{-}\lim_{\varepsilon\to 0} I_\varepsilon(\psi)=\int_\Sigma \mathcal{Q}W_0(\nabla\psi(x))dx.$$

Although the Le Dret-Raoult theorem is compatible neither with (w-DC) nor (s-DC) it established a suitable variational framework to deal with dimensional reduction problems : it is the point of departure of many works on the subject.

2.4. $\Gamma(\pi)$**-convergence of** I_ε**: "weak-Determinant Constraint".**
By using the Le Dret-Raoult theorem we can prove the following result.

Theorem C (see [1, 5])**.**
Assume that

(D) $\exists\alpha,\beta>0\ \forall F\in\mathbb{M}^{3\times 3}\ \big(|\mathrm{det}F|\geq\alpha\Rightarrow W(F)\leq\beta(1+|F|^p)\big)$.

Then

$$\forall\psi\in W^{1,p}(\Sigma;\mathbb{R}^3)\quad \Gamma(\pi)\text{-}\lim_{\varepsilon\to 0}I_\varepsilon(\psi)=\int_\Sigma \mathcal{Q}W_0(\nabla\psi(x))dx.$$

Outline of the proof. ▶ As the $\Gamma(\pi)$-limit is stable by substituting I_ε by its relaxed functional $\overline{I}_\varepsilon$, i.e., $\overline{I}_\varepsilon:W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3)\to[0,+\infty]$ given by

$$I_\varepsilon(\phi):=\inf\left\{\liminf_{n\to+\infty}I_\varepsilon(\phi_n):\phi_n\stackrel{L^p}{\to}\phi\right\}=\frac{1}{\varepsilon}\inf\left\{\liminf_{n\to+\infty}\int_{\Sigma_\varepsilon}W(\nabla\phi_n)dxdx_3:\phi_n\stackrel{L^p}{\to}\phi\right\},$$

it suffices to prove that

$$\forall\psi\in W^{1,p}(\Sigma;\mathbb{R}^3)\quad \Gamma(\pi)\text{-}\lim_{\varepsilon\to 0}\overline{I}_\varepsilon(\psi)=\int_\Sigma \mathcal{Q}W_0(\nabla\psi(x))dx.$$

▶ As W satisfies (D) it is p-ample (see §1.6), and so by Theorem A-B we have

$$\forall\varepsilon>0\ \forall\phi\in W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3)\quad \overline{I}_\varepsilon(\phi)=\frac{1}{\varepsilon}\int_{\Sigma_\varepsilon}\mathcal{Q}W(\nabla\phi(x,x_3))dxdx_3$$

with $\mathcal{Q}W=\mathcal{Z}_\infty W$ (which is of p-polynomial growth and so continuous by the property (2) of Fonseca).

▶ Applying the Le Dret-Raoult theorem we deduce that

$$\forall\psi\in W^{1,p}(\Sigma;\mathbb{R}^3)\quad \Gamma(\pi)\text{-}\lim_{\varepsilon\to 0}\overline{I}_\varepsilon(\psi)=\int_\Sigma \mathcal{Q}[\mathcal{Q}W]_0(\nabla\psi(x))dx$$

with $[\mathcal{Q}W]_0:\mathbb{M}^{3\times 2}\to[0,+\infty]$ given by

$$[\mathcal{Q}W]_0(\xi):=\inf_{\zeta\in\mathbb{R}^3}\mathcal{Q}W(\xi\mid\zeta).$$

▶ Finally, we prove that $\mathcal{Q}[\mathcal{Q}W]_0=\mathcal{Q}W_0$, and the proof is complete. □

Theorem C highlights the fact that the concept of p-amplitude has a "nice" behavior with respect to the $\Gamma(\pi)$-convergence. More generally, let $\{\pi_\varepsilon\}_\varepsilon$ be a family of L^p-continuous maps π_ε from $W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^m)$ to $W^{1,p}(\Sigma;\mathbb{R}^m)$, where $\Sigma_\varepsilon\subset\mathbb{R}^N$ (resp. $\Sigma\subset\mathbb{R}^k$ with $k\in\mathbb{N}^*$) is a bounded open set, let $\{W_\varepsilon\}_\varepsilon$ be an uniformly p-coercive family of measurable integrands $W_\varepsilon:\mathbb{M}^{m\times N}\to[0,+\infty]$ and, for each $\varepsilon>0$, let $I_\varepsilon,\mathcal{Q}I_\varepsilon:W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^m)\to[0,+\infty]$ be respectively defined by

- ♦ $I_\varepsilon(\phi):=\displaystyle\int_{\Sigma_\varepsilon}W_\varepsilon(\nabla\phi(x))dx$;
- ♦ $\mathcal{Q}I_\varepsilon(\phi):=\displaystyle\int_{\Sigma_\varepsilon}\mathcal{Q}W_\varepsilon(\nabla\phi(x))dx$.

The following theorem says that the $\Gamma(\pi)$-limit is stable by substituting I_ε by $\mathcal{Q}I_\varepsilon$ whenever every W_ε is p-ample.

Theorem (see [5])**.**
Assume that:

- ♦ $\forall\varepsilon>0$ W_ε *is* p*-ample*;
- ♦ $\exists I_0:W^{1,p}(\Sigma;\mathbb{R}^m)\to[0,+\infty]$ $\Gamma(\pi)\text{-}\lim_{\varepsilon\to 0}\mathcal{Q}I_\varepsilon=I_0$.

Then $\Gamma(\pi)\text{-}\lim_{\varepsilon\to 0}I_\varepsilon=I_0$.

Proof. As every W_ε is p-ample, from Theorem A-B we deduce that $\overline{I}_\varepsilon = \mathcal{Q}I_\varepsilon$ for all $\varepsilon > 0$. On the other hand, as every π_ε is L^p-continuous, it is easy to see that $\Gamma(\pi)$-$\liminf_{\varepsilon\to 0} I_\varepsilon = \Gamma(\pi)$-$\liminf_{\varepsilon\to 0}\overline{I}_\varepsilon$ and $\Gamma(\pi)$-$\limsup_{\varepsilon\to 0} I_\varepsilon = \Gamma(\pi)$-$\limsup_{\varepsilon\to 0}\overline{I}_\varepsilon$, and the theorem follows. □

2.5. $\Gamma(\pi)$-convergence of I_ε: "strong-Determinant Constraint".

The following theorem gives an answer to $(\mathcal{P}_2)$ in the framework of nonlinear elasticity (it is consistent with (s-DC)) in the same spirit as the theorem of Ball in 1977 (see [7]). It is the result of several works on the subject: mainly, the attempt of Percivale in 1991 (see [22]), the rigorous answer to $(\mathcal{P}_2)$ by Le Dret and Raoult in the p-polynomial growth case (see [20, 21]) and especially the substantial contributions of Ben Belgacem (see [8, 9, 10]).

Theorem D (see [3, 5])**.**
Assume that:

- (D_0) W *is continuous*;
- (D_1) $W(F) = +\infty \iff \det F \le 0$;
- (D_2) $\forall \delta > 0\ \exists c_\delta > 0\ \forall F \in \mathbb{M}^{3\times 3}\big(\det F \ge \delta \Rightarrow W(F) \le c_\delta(1 + |F|^p)\big)$.

Then

$$\forall \psi \in W^{1,p}(\Sigma;\mathbb{R}^3) \quad \Gamma(\pi)\text{-}\lim_{\varepsilon\to 0} I_\varepsilon(\psi) = \int_\Sigma \mathcal{Q}W_0(\nabla\psi(x))dx.$$

Outline of the proof. ► It is easy to see that if W satisfies (D_0), (D_1) and (D_2) then:

- (P_0) W_0 is continuous;
- (P_1) $\forall \alpha > 0\ \exists \beta_\alpha > 0\ \forall \xi \in \mathbb{M}^{3\times 2}\big(|\xi_1 \wedge \xi_2| \ge \alpha \Rightarrow W_0(\xi) \le \beta_\alpha(1 + |\xi|^p)\big)$.

In particular, W_0 satisfies (P) since clearly (P_1) implies (P).

► Let $\mathcal{I}, \overline{\mathcal{I}}, \overline{\mathcal{I}}_{\mathrm{diff}_*} : W^{1,p}(\Sigma;\mathbb{R}^3) \to [0,+\infty]$ be respectively defined by:

- ♦ $\mathcal{I}(\psi) := \displaystyle\int_\Sigma W_0(\nabla\psi(x))dx$;
- ♦ $\overline{\mathcal{I}}(\psi) := \inf\left\{\liminf\limits_{n\to+\infty}\mathcal{I}(\psi_n) : \psi_n \xrightarrow{L^p} \psi\right\}$;
- ♦ $\overline{\mathcal{I}}_{\mathrm{diff}_*}(\psi) := \inf\left\{\liminf\limits_{n\to+\infty}\mathcal{I}(\psi_n) : C^1_*(\overline{\Sigma};\mathbb{R}^3) \ni \psi_n \xrightarrow{L^p} \psi\right\}$,

where $C^1_*(\overline{\Sigma};\mathbb{R}^3)$ is the set of C^1-immersions from $\overline{\Sigma}$ to $\mathbb{R}^3$, i.e.,

$$C^1_*(\overline{\Sigma};\mathbb{R}^3) := \Big\{\psi \in C^1(\overline{\Sigma};\mathbb{R}^3) : \forall x \in \overline{\Sigma}\ \partial_1\psi(x) \wedge \partial_2\psi(x) \neq 0\Big\}.$$

As W_0 satisfies (P), by Corollary 1 we have

$$\forall \psi \in W^{1,p}(\Sigma;\mathbb{R}^3) \quad \overline{\mathcal{I}}(\psi) = \int_\Sigma \mathcal{Q}W_0(\nabla\psi(x))dx.$$

On the other hand, we can prove the following two lemmas.

Lemma. $\overline{\mathcal{I}} \le \Gamma(\pi)\text{-}\liminf\limits_{\varepsilon\to 0} I_\varepsilon$.

Lemma. *If* (D_0), (D_1) *and* (D_2) *hold then* $\Gamma(\pi)\text{-}\limsup\limits_{\varepsilon\to 0} I_\varepsilon \le \overline{\mathcal{I}}_{\mathrm{diff}_*}$.

Hence, it suffices to prove that $\overline{\mathcal{I}}_{\mathrm{diff}_*} \le \overline{\mathcal{I}}$.

► Let $\overline{\mathcal{I}}_{\mathrm{aff}_{\mathrm{li}}}, \mathcal{R}\mathcal{I}, \overline{\mathcal{R}\mathcal{I}}, \overline{\mathcal{R}\mathcal{I}}_{\mathrm{aff}_{\mathrm{li}}} : W^{1,p}(\Sigma;\mathbb{R}^3) \to [0,+\infty]$ be respectively defined by:

- ♦ $\overline{\mathcal{I}}_{\mathrm{aff}_{\mathrm{li}}}(\psi) := \inf\left\{\liminf\limits_{n\to+\infty}\mathcal{I}(\psi_n) : \mathrm{Aff}_{\mathrm{li}}(\Sigma;\mathbb{R}^3) \ni \psi_n \xrightarrow{L^p} \psi\right\}$;
- ♦ $\mathcal{R}\mathcal{I}(\psi) := \displaystyle\int_\Sigma \mathcal{R}W_0(\nabla\psi(x))dx$;

- $\overline{\mathcal{RI}}(\psi) := \inf\left\{\liminf\limits_{n\to+\infty} \mathcal{RI}(\psi_n) : \psi_n \stackrel{L^p}{\to} \psi\right\}$;
- $\overline{\mathcal{RI}}_{\mathrm{aff_{li}}}(\psi) := \inf\left\{\liminf\limits_{n\to+\infty} \mathcal{RI}(\psi_n) : \mathrm{Aff_{li}}(\Sigma;\mathbb{R}^3) \ni \psi_n \stackrel{L^p}{\to} \psi\right\}$

with $\mathrm{Aff_{li}}(\Sigma;\mathbb{R}^3) := \big\{\psi \in \mathrm{Aff}(\Sigma;\mathbb{R}^3) : \psi \text{ is locally injective}\big\}$. As $\overline{\mathcal{RI}} \le \overline{\mathcal{I}}$, a way for proving $\overline{\mathcal{I}}_{\mathrm{diff}_*} \le \overline{\mathcal{I}}$ is to establish the following three inequalities:

- $\overline{\mathcal{I}}_{\mathrm{diff}_*} \le \overline{\mathcal{I}}_{\mathrm{aff_{li}}}$;
- $\overline{\mathcal{I}}_{\mathrm{aff_{li}}} \le \overline{\mathcal{RI}}_{\mathrm{aff_{li}}}$;
- $\overline{\mathcal{RI}}_{\mathrm{aff_{li}}} \le \overline{\mathcal{RI}}$.

The first inequality follows by using the fact that W_0 satisfies (P_0) and (P_1) together with the following lemma.

Lemma (Ben Belgacem-Bennequin [8] 1996, see also [5])**.**
For all $\psi \in \mathrm{Aff_{li}}(\Sigma;\mathbb{R}^3)$ there exists $\{\psi_n\}_{n\ge1} \subset C^1_*(\overline{\Sigma};\mathbb{R}^3)$ such that:

- $\psi_n \stackrel{W^{1,p}}{\to} \psi$;
- $\exists\delta > 0\ \forall x \in \overline{\Sigma}\ \forall n \ge 1\ |\partial_1\psi_n(x) \wedge \partial_2\psi_n(x)| \ge \delta$.

The second inequality is obtained by exploiting the Kohn-Strang representation of $\mathcal{R}W_0$ (see [8], see also [5]). Finally, we establish the next inequality by combining the following two lemmas.

Lemma (Ben Belgacem [8] 1996, see also [5])**.**
If W_0 satisfies (P_0) and (P_1) then:

- *$\mathcal{R}W_0$ is continuous;*
- $\exists c > 0\ \forall \xi \in \mathbb{M}^{3\times2}\ \mathcal{R}W_0(\xi) \le c(1+|\xi|^p)$.

Lemma (Gromov-Èliašberg [18] 1971, see also [5])**.**
$\mathrm{Aff_{li}}(\Sigma;\mathbb{R}^3)$ *is strongly dense in* $W^{1,p}(\Sigma;\mathbb{R}^3)$. □

Question. *Try to simplify the proof of Theorem D as follows: first, approximate W satisfying (D_0), (D_1) and (D_2) or maybe weaker conditions compatible with (s-DC) by a supremum of p-ample integrands W_δ satisfying (D) with $\alpha, \beta > 0$ which can depend on δ, then, apply Theorem C to each W_δ, and finally, pass to the limit as δ goes to zero.*

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